MEASURE SPACES AND INTEGRATION OF MEASURABLE FUNCTIONS

Measurable Sets

A **σ-algebra** is a family of sets $\mathcal{B} \subseteq \mathscr{P}(X)$ such that

- $X \in \mathcal{B}$
- $B \in \mathcal{B} \implies X \setminus B \in \mathcal{B}$
- $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B} \implies \bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$

A pair $(X, \mathcal{B})$ is a **measurable space** if X is a set and $\mathcal{B} \subseteq \mathscr{P}(X)$ is a σ-algebra. Elements of $\mathcal{B}$ are the **measurable sets**.

Measurable Functions

Given $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ measurable spaces, a function $f : X \to Y$ is **measurable** if $C \in \mathcal{C} \implies f^{-1}(C) \in \mathcal{B}$.

If $(X, \mathcal{B})$ is a measurable space and (Y, τ) is a topological space, a function $f : X \to Y$ is **measurable** if $V \in \tau \implies f^{-1}(V) \in \mathcal{B}$.

Measurability criterion. Let $(X, \mathcal{B})$ be a measurable space and $f : X \to [-\infty, \infty]$. TFAE:

(i) f is measurable
(ii) for every $c \in \mathbb{R}$, $\{f > c\} \in \mathcal{B}$
(iii) for every $c \in \mathbb{R}$, $\{f \geq c\} \in \mathcal{B}$
(iv) for every $c \in \mathbb{R}$, $\{f < c\} \in \mathcal{B}$
(v) for every $c \in \mathbb{R}$, $\{f \leq c\} \in \mathcal{B}$

Measures

Let $(X, \mathcal{B})$ be a measurable space. A **measure** is a function $\mu : \mathcal{B} \to [0, \infty]$ satisfying

- $\mu(\emptyset) = 0$
- for $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}$ pairwise disjoint, $\mu\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

Basic properties of measures.

- monotone: $A \subseteq B \implies \mu(A) \leq \mu(B)$
- countably subadditive: $\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$
- continuous from below: if $E_1 \subseteq E_2 \subseteq \dots$, then $\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n)$
- continuous from above: if $E_1 \supseteq E_2 \supseteq \dots$ and $\mu(E_1) < \infty$, then $\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n)$

Integration of Simple Functions

A **simple function** is a measurable function taking finitely many values or, equivalently, a linear combination of indicator functions of measurable sets $s = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}$.

The **integral of s with respect to μ** is

$$\int_X s\; d\mu = \sum_{j=1}^{n} c_j \mu(E_j).$$

Integration of Nonnegative Functions

Let $f : X \to [0, \infty]$. The **integral of f with respect to** μ is $\int_X f\; d\mu = \sup\{\int_X s\; d\mu : 0 \leq s \leq f, s \text{ simple}\}$.

Monotone Convergence Theorem. If $0 \leq f_1 \leq f_2 \leq \cdots \to f$, then

$$\int_X f\; d\mu = \lim_{n\to\infty} \int_X f_n\; d\mu.$$

Fatou's Lemma. If $f_n \geq 0$, then

$$\int_X \liminf_{n\to\infty} f_n\; d\mu \leq \liminf_{n\to\infty} \int_X f_n\; d\mu.$$

Integration of Complex Functions

Decompose $f = (f_1 - f_2) + i(f_3 - f_4)$ with $f_j \geq 0$. Then f is **integrable** if $\int_X |f|\; d\mu < \infty$ (equivalently, $\int_X f_j\; d\mu < \infty$ for every $j \in \{1, 2, 3, 4\}$), and the **integral of f with respect to** μ is

$$\int_X f\; d\mu = \left(\int_X f_1\; d\mu - \int_X f_2\; d\mu\right) + i\left(\int_X f_3\; d\mu - \int_X f_4\; d\mu\right).$$

Basic properties of the integral.

- triangle inequality: $|\int_X f\; d\mu| \leq \int_X |f|\; d\mu$
- linearity: $\int_X (cf+g)\; d\mu = c\int_X f\; d\mu + \int_X g\; d\mu$

Dominated Convergence Theorem. If $f_n \to f$ a.e. and $|f_n| \leq g \in L^1(\mu)$, then $f_n \to f$ in L^1. In particular,

$$\int_X f\; d\mu = \lim_{n\to\infty} \int_X f_n\; d\mu.$$